UNIQUE WORLD EDUCATION

HANDBOOK OF MATHEMATICS FORMULAE

GRADE : 12 CBSE

SUBJECT : MATHEMATICS

INDEX

RELATIONS AND FUNCTIONS

1
CHAPTER

Relations:

Let A and B be two non-empty sets, then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a, b) \in R$. Then we say that a is related to b by the relation R and write it as $a R b$. If $(a,b) \in R$, we write it as $a\ R\ b$.

Total number of relations:

Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. so total number of relations from A to B is 2^{mn}.

Domain and range of a relation:

Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

Relation on a set:

Let A be a non-void set. Then, a relation from A to itself *i.e.* a subset of $A \times A$ is called a relation on set A.

Reflexive relation:

A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

Symmetric relation:

A relation R on a set A is said to be a symmetric relation *iff*

$$(a, b) \in R \Rightarrow (b, a) \in R \text{ for all } a, b \in A$$

Transitive relation:

Let A be any set. A relation R on set A is said to be a transitive relation *iff*

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e., aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

Equivalence relation:

A relation R on a set A is said to be an equivalence relation on A *iff*

(i) It is reflexive *i.e.* $(a, a) \in R$ for all $a \in A$

(ii) It is symmetric *i.e.* $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$

(iii) It is transitive *i.e.* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Equivalence Classes of An Equivalence Relation:

Let R be equivalence relation in $A(\neq \varphi)$. Let $a \in A$. Then the equivalence class of a, denoted by $[a]$ or $\{\overline{a}\}$ is defined as the set of all those points of A which are related to a under the relation R. Thus $[a] = \{x \in A : x\, R\, a\}$.

Definition of Function:

Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write $f : X \to Y$ where $y = f(x), x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under f (or x is the pre image of y).

Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X.

One-One Function (injection):

A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B. Thus,

$f : A \to B$ is one-one $\Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$

Many-One Function:

A function $f : A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

Thus, $f : A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

Onto Function (Surjection):

A function $f : A \to B$ is onto if each element of B has its pre-image in A. Therefore, if $f^{-1}(y) \in A, \forall y \in B$ then function is onto. In other words, Range of f = Co-domain of f.

One-One Onto Function (Bijection):

A function $f : A \to B$ is a bijection if it is one-one as well as onto. In other words, a function $f : A \to B$ is a bijection if

(i) It is one-one i.e., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) It is onto i.e., for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Clearly, f is a bijection since it is both injective as well as surjective.

Composite Function

If $f : A \to B$ and $g : B \to C$ are two function then the composite function of f and g, $gof\ A \to C$ will be defined as $gof(x) = g[f(x)], \forall x \in A$

Inverse Function:

If $f : A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1} : B \to A$ which associates each element $b \in B$ with element $a \in A$, such that $f(a) = b$, is called the inverse function of the function $f : A \to B$

$$f^{-1} : B \to A,\ f^{-1}(b) = a \Rightarrow f(a) = b$$

INVERSE TRIGONOMETRIC FUNCTIONS

2

CHAPTER

The Inverse of a Function:

The inverse of a function $f : A \to B$ exists if f is one-one onto *i.e.*, a bijection and is given by $f(x) = y \Rightarrow f^{-1}(y) = x$.

Meaning of inverse function:

(i) $\sin\theta = x \Rightarrow \sin^{-1}x = \theta$ **(ii)** $\cos\theta = x \Rightarrow \cos^{-1}x = \theta$

(iii) $\tan\theta = x \Rightarrow \tan^{-1}x = \theta$ **(iv)** $\cot\theta = x \Rightarrow \cot^{-1}x = \theta$

(v) $\sec\theta = x \Rightarrow \sec^{-1}x = \theta$ **(vi)** $\text{cosec}\,\theta = x \Rightarrow \text{cosec}^{-1}x = \theta$

Domain and Range of inverse functions:

Function	Domain (D)	Range (R)
$\sin^{-1}x$	$-1 \le x \le 1$ or $[-1,1]$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$-1 \le x \le 1$ or $[-1,1]$	$0 \le \theta \le \pi$ or $[0,\pi]$
$\tan^{-1}x$	$-\infty < x < \infty$ *i.e.*, $x \in R$ or $(-\infty,\infty)$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	$-\infty < x < \infty$ *i.e.*, $x \in R$ or $(-\infty,\infty)$	$0 < \theta < \pi$ or $(0,\pi)$
$\sec^{-1}x$	$x \le -1, x \ge 1$ or $(-\infty,-1] \cup [1,\infty)$	$\theta \ne \frac{\pi}{2}, 0 \le \theta \le \pi$ or $\left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$
$\text{cosec}^{-1}x$	$x \le -1, x \ge 1$ or $(-\infty,-1] \cup [1,\infty)$	$\theta \ne 0, \frac{-\pi}{2}2 \le \theta \le \frac{\pi}{2}$ or $\left[-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right]$

Formulas 1:

(1) $\sin^{-1}(\sin\theta) = \theta$, Provided that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, **(2)** $\cos^{-1}(\cos\theta) = \theta$, Provided that $0 \le \theta \le \pi$

(3) $\tan^{-1}(\tan\theta) = \theta$, Provided that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, **(4)** $\cot^{-1}(\cot\theta) = \theta$, Provided that $0 < \theta < \pi$

(5) $\sec^{-1}(\sec\theta) = \theta$, Provided that $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

(6) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, Provided that $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$

Formulas 2:

$\sin(\sin^{-1} x) = x$, Provided that $-1 \le x \le 1$, $\cos(\cos^{-1} x) = x$, Provided that $-1 \le x \le 1$

$\tan(\tan^{-1} x) = x$, Provided that $-\infty < x < \infty$ $\cot(\cot^{-1} x) = x$, Provided that $-\infty < x < \infty$

$\sec(\sec^{-1} x) = x$, Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

$\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

Formulas 3:

(1) $\sin^{-1}(-x) = -\sin^{-1} x,\ x \in [-1, 1]$ **(2)** $\cos^{-1}(-x) = \pi - \cos^{-1} x,\ x \in [-1, 1]$

(3) $\tan^{-1}(-x) = -\tan^{-1} x,\ x \in R$ **(4)** $\cot^{-1}(-x) = \pi - \cot^{-1} x,\ x \in R$

(5) $\sec^{-1}(-x) = \pi - \sec^{-1} x,\ |x| \ge 1$ **(6)** $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x,\ |x| \ge 1$

Formulas 4:

(1) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},\ x \in [-1, 1]$ **(2)** $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ x \in R$

(3) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2},\ |x| \ge 1$ **(4)** $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x,\ x \ge 1$ or $x \le 1$

(5) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x,\ x \ge 1$ or $x \le 1$ **(5)** $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x,\ x \ge 0$

Principal Values for Inverse Trigonometric Functions:

Principal values for $x \ge 0$	Principal values for $x < 0$
$0 \le \sin^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \sin^{-1} x < 0$
$0 \le \cos^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \le \pi$
$0 \le \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 < \cot^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \le \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \le \pi$
$0 < \operatorname{cosec}^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \operatorname{cosec}^{-1} x < 0$

Formulae for Sum and Difference of Inverse Trigonometric Function

(1) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, **(2)** $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(3) $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, **(4)** $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(5) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

MATRICES

3 CHAPTER

Matrix:

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, enclosed by [ ]is called a matrix. The numbers are called the elements of the matrix or entries in the matrix. A matrix is represented by capital letters *A, B, C etc.* and its elements by small letters *a, b, c, x, y etc.*

Order of a Matrix:

A matrix having *m* rows and *n* columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as 'an *m* by *n* matrix). A matrix *A* of order $m \times n$ is usually written in the following manner

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots a_{1j} & \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots a_{2j} & \ldots a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots a_{ij} & \ldots a_{in} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots a_{mj} & \ldots a_{mn} \end{bmatrix} \text{or } A = [a_{ij}]_{m\times n}, \text{ where } \begin{matrix} i = 1, 2, \ldots\ldots m \\ j = 1, 2, \ldots\ldots n \end{matrix}$$

Here a_{ij} denotes the element of ith row and jth column.

Equality of Matrices

Two matrix *A* and *B* are said to be equal matrix if they are of same order and their corresponding elements are equal

TYPES OF MATRICES

Row matrix: A matrix is said to be a row matrix or row vector if it has only one row and any number of columns.

Column matrix: A matrix is said to be a column matrix or column vector if it has only one column and any number of rows.

Singleton matrix: If in a matrix there is only one element then it is called singleton matrix.

Null or zero matrix: If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by *O*. Thus $A = [a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij} = 0$ for all *i* and *j*.

Square matrix: If number of rows and number of columns in a matrix are equal, then it is called a square matrix. Thus $A = [a_{ij}]_{m\times n}$ is a square matrix if $m = n$.

If $m \neq n$ then matrix is called a rectangular matrix.

Diagonal matrix: If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix. Thus a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

Identity matrix: A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix. Thus, the square matrix $A = [a_{ij}]$ is an identity matrix,

if $a_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$

We denote the identity matrix of order *n* by I_n.

Scalar matrix: A square matrix whose all non diagonal elements are zero and diagonal elements are equal is called a scalar matrix. Thus, if $A=[a_{ij}]$ is a square matrix and $a_{ij}=\begin{cases}\alpha, \text{if } i=j\\ 0, \text{if } i\neq j\end{cases}$, then A is a scalar matrix.

Addition and Subtraction of Matrices:

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order then their sum $A+B$ is a matrix whose each element is the sum of corresponding elements. *i.e.* $A+B=[a_{ij}+b_{ij}]_{m\times n}$

Similarly, their subtraction $A-B$ is defined as $A-B=\left[a_{ij}-b_{ij}\right]_{m\times n}$

Properties of matrix addition: If *A, B* and *C* are matrices of same order, then

(i) $A+B=B+A$ (Commutative law)

(ii) $(A+B)+C=A+(B+C)$ (Associative law)

(iii) $A+O=O+A=A$, where O is zero matrix which is additive identity of the matrix.

(iv) $A+(-A)=0=(-A)+A$, where $(-A)$ is obtained by changing the sign of every element of *A,* which is additive inverse of the matrix.

Scalar Multiplication of Matrices:

Let $A=[a_{ij}]_{m\times n}$ be a matrix and k be a number, then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by kA.

Thus, if $A=[a_{ij}]_{m\times n}$, then $kA=Ak=[ka_{ij}]_{m\times n}$.

Properties of scalar multiplication:

If A, B are matrices of the same order and λ, μ are any two scalars then

(i) $\lambda(A+B)=\lambda A+\lambda B$ **(ii)** $(\lambda+\mu)A=\lambda A+\mu A$

(iii) $\lambda(\mu A)=(\lambda\mu A)=\mu(\lambda A)$ **(iv)** $(-\lambda A)=-(\lambda A)=\lambda(-A)$

Multiplication of Matrices

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in *B* (post multiplier).Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ are two matrices of order $m\times n$ and $n\times p$ respectively, then their product AB is of order $m\times p$ and is defined as

$$(AB)_{ij}=\sum_{r=1}^{n} a_{ir}b_{rj}=[a_{i1}a_{i2}...a_{in}]\begin{bmatrix}b_{1j}\\ b_{2j}\\ \vdots\\ b_{nj}\end{bmatrix}=(i^{\text{th}}\text{ row of A})\ (j^{\text{th}}\text{ column of } B) \qquad ...\text{(i)}$$

Properties of matrix multiplication:

If *A*, *B* and *C* are three matrices such that their product is defined, then

(i) $AB\neq BA$ (Generally not commutative)

(ii) $(AB)C=A(BC)$ (Associative Law)

(iii) $IA=A=AI$ where I is identity matrix for matrix multiplication

(iv) $A(B+C)=AB+AC$ (Distributive law)

(v) If $AB=0$ It does not mean that $A=0$ or $B=0$, again product of two non zero matrix may be a zero matrix.

Transpose of a Matrix:

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of Matrix A and is denoted by A^T or A'.

From the definition it is obvious that if order of A is $m \times n$, then order of A^T is $n \times m$

Properties of transpose: Let A and B be two matrices then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$, A and B being of the same order

(iii) $(kA)^T = kA^T$, k be any scalar (real or complex)

(iv) $(AB)^T = B^T A^T$, A and B being conformable for the product AB

(v) $(A_1 A_2 A_3 .....A_{n-1} A_n)^T = A_n{}^T A_{n-1}{}^T ....A_3{}^T A_2{}^T A_1{}^T$

(vi) $I^T = I$

SPECIAL TYPES OF MATRICES

Symmetric Matrix: A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all i, j or $A^T = A$

Skew-symmetric matrix: A square matrix $A = [a_{ij}]$ is called skew- symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$.

Properties of symmetric and skew-symmetric matrices:

(i) If A is a square matrix, then $A + A^T, AA^T, A^T A$ are symmetric matrices, while $A - A^T$ is skew-symmetric matrix.

(ii) Every square matrix A can uniquely be expressed as sum of a symmetric and skew-symmetric matrix *i.e.*

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right].$$

Elementary Transformations or Elementary Operations of a Matrix

Interchange of any two rows (columns):

If i^{th} row (column) of a matrix is interchanged with the i^{th} row (column), it will be denoted by $R_i \leftrightarrow R_j (C_i \leftrightarrow C_j)$

Multiplying all elements of a row (column) of a matrix by a non-zero scalar:

If the elements of i^{th} row (column) are multiplied by a non-zero scalar k, it will be denoted by $R_i \to R_i(k), [C_i \to C_i(k)]$ or $R_i \to kR_i, [C_i \to kC_i]$

Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar *k*:

If k times the elements of j^{th} row (column) are added to the corresponding elements of the i^{th} row (column), it will be denoted by $R_i \to R_i + kR_j (C_i \to C_i + kC_j)$

DETERMINANTS

4
CHAPTER

Determinants:

Second order determinant:

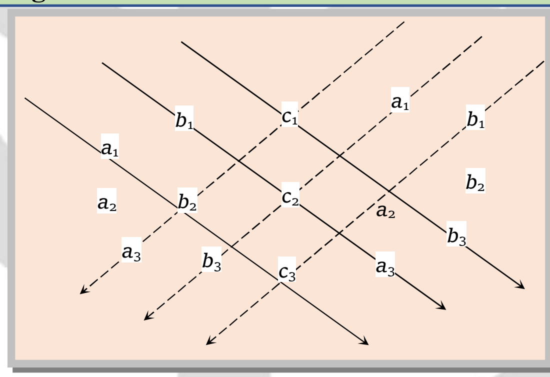
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 .$$

Third order determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Short cut method or Sarrus diagram method:

To find the value of third order determinant, following method is also useful



Taking product of R.H.S. diagonal elements positive and L.H.S. diagonal elements negative and adding them. We get the value of determinant as

$$= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3$$

PROPERTIES OF DETERMINANTS

Property-1: The value of determinant remains unchanged, if the rows and the columns are interchanged.

If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Then $D' = D$, D and D' are transpose of

Property-2: If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = -D$

Property-3: If a determinant has two rows (or columns) identical, then its value is zero.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$. Then, $D = 0$

Property-4: If all the elements of any row (or column) be multiplied by the same number, then the value of determinant is multiplied by that number.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = kD$

Property-5: If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the determinants.

$$e.g., \quad \begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property-6: The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column)

$e.g.,$ $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 - na_1 & b_3 - nb_1 & c_3 - nc_1 \end{vmatrix}$. Then $D' = D$

MINORS AND COFACTORS

Minor of an element: If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}

Consider the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of minors $M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}$,

where M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, M_{12} = minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ etc.

Cofactor of an element: The cofactor of an element a_{ij} (*i.e.* the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij} or A_{ij} or F_{ij}.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of cofactors is $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$ where, $C_{ij} = (-1)^{i+j} M_{ij}$

Adjoint of a Square Matrix:

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by *adj* A

Thus, $adj\ A = [C_{ij}]^T \Rightarrow (adj\ A)_{ij} = C_{ji}$ = cofactor of a_{ji} in A.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\ A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$;

Inverse of a Matrix:

A non-singular square matrix of order *n* is invertible if there exists a square matrix *B* of the same order such that $AB = I_n = BA$. Then we say that the inverse of *A* is *B* and we write $A^{-1} = B$

The inverse of A is given by $A^{-1} = \frac{1}{|A|}.adj\ A$

Properties of inverse matrix:

If *A* and *B* are invertible matrices of the same order, then

(i) $(A^{-1})^{-1} = A$ **(ii)** $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$ **(iv)** $(A^k)^{-1} = (A^{-1})^k, k \in N$ [In particular $(A^2)^{-1} = (A^{-1})^2$]

Application of Determinants in Co-Ordinate Geometry

Area of Triangle: Area of triangle whose vertices are (x_r, y_r); $r = 1, 2, 3$ is

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Collinearity of three points:

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

System of Simultaneous Linear Equations:

Consider the following system of *m* linear equations in *n* unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ ... \quad ... \quad ... \quad ... \quad ... \\ ... \quad ... \quad ... \quad ... \quad ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned}$$

The system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ or } AX = B,$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

The $m \times n$ matrix A is called the coefficient matrix of the system of linear equations.

Matrix method for solving Simultaneous linear equations:

The system of equations $AX = B$ has a solution given by: $X = A^{-1}B$, where $A^{-1} = \frac{1}{|A|}adj\ A$

CONTINUITY AND DIFFERENTIABILITY

5 CHAPTER

Formulas from Limits

(i) $\lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{x}{\sin x}$

(ii) $\lim_{x\to 0}\frac{\tan x}{x} = 1 = \lim_{x\to 0}\frac{x}{\tan x}$

(iii) $\lim_{x\to 0}\cos x = 1$

(iv) $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

(v) $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$

(vi) $\lim_{x\to 0}\frac{a^x - 1}{x} = \log_e a$

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain iff $\lim_{x\to a} f(x) = f(a)$. *i.e.* a function $f(x)$ is continuous at $x = a$ if and only if it satisfies the following three conditions :

(1) $f(a)$ exists. ('a' lies in the domain of f)

(2) $\lim_{x\to a} f(x)$ exist *i.e.* $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ or R.H.L. = L.H.L.

(3) $\lim_{x\to a} f(x) = f(a)$ (limit equals the value of function).

Continuity from Left and Right:

Function $f(x)$ is said to be

(1) Left continuous at $x = a$ if $\lim_{x\to a-0} f(x) = f(a)$

(2) Right continuous at $x = a$ if $\mathrm{Lim}_{x\to a+0} f(x) = f(a)$.

Thus a function $f(x)$ is continuous at a point $x = a$ if it is left continuous as well as right continuous at $x = a$.

Algebra of Continuous Functions:

Let $f(x)$ and $g(x)$ be two continuous functions at $x = a$. Then

(i) $cf(x)$ is continuous at $x = a$, where c is any constant

(ii) $f(x) \pm g(x)$ is continuous at $x = a$.

(iii) $f(x).\ g(x)$ is continuous at $x = a$.

(iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

(v) If the function $u = f(x)$ is continuous at the point $x = a$, and the function $y = g(u)$ is continuous at the point $u = f(a)$, then the composite function $y = (gof)(x) = g(f(x))$ is continuous at the point $x = a$.

Discontinuous function:

A function 'f' which is not continuous at a point $x = a$ in its domain is said to be discontinuous there at. The point 'a' is called a point of discontinuity of the function.

The discontinuity may arise due to any of the following situations.

(i) $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or both may not exist

(ii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.

DIFFERENTIATION:

Derivative at a Point:

The derivative of a function at a point $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ (provided the limit exists and is finite)}$$

The above definition of derivative is also called derivative by first principle.

SOME STANDARD DERIVATIVES:

(1) $\frac{d}{dx} x^n = nx^{n-1}, x \in R, n \in R, x > 0$

(2) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

(3) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$

(4) $\frac{d}{dx} \sin x = \cos x$

(5) $\frac{d}{dx} \cos x = -\sin x$

(6) $\frac{d}{dx} \tan x = \sec^2 x$

(7) $\frac{d}{dx} \sec x = \sec x \tan x$

(8) $\frac{d}{dx} \text{cosec } x = -\text{cosec } x \cot x$

(9) $\frac{d}{dx} \cot x = -\text{cosec}^2 x$

(10) $\frac{d}{dx} \log x = \frac{1}{x}$, for $x > 0$

(11) $\frac{d}{dx} e^x = e^x$

(12) $\frac{d}{dx} a^x = a^x \log a$, for $a > 0$

(13) $\frac{d}{dx} \log_a x = \frac{1}{x \log a}$,

(14) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$

(15) $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$

(16) $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$

(17) $\frac{d}{dx} \text{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2 - 1}}$

(18) $\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$

(19) $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1 + x^2}$

Differentiation by inverse trigonometrical substitution: For Inverse trigonometrical functions following substitution should be remembered

Some suitable substitutions

S. N.	Function	Substitution	S. N.	Function	Substitution
(i)	$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $a\cos\theta$	(ii)	$\sqrt{x^2+a^2}$	$x=a\tan\theta$ or $a\cot\theta$
(iii)	$\sqrt{x^2-a^2}$	$x=a\sec\theta$ or $a\operatorname{cosec}\theta$	(iv)	$\sqrt{\frac{a-x}{a+x}}$	$x=a\cos 2\theta$
(v)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2=a^2\cos 2\theta$	(vi)	$\sqrt{ax-x^2}$	$x=a\sin^2\theta$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x=a\tan^2\theta$	(viii)	$\sqrt{\frac{x}{a-x}}$	$x=a\sin^2\theta$
(ix)	$\sqrt{(x-a)(x-b)}$	$x=a\sec^2\theta-b\tan^2\theta$	(x)	$\sqrt{(x-a)(b-x)}$	$x=a\cos^2\theta+b\sin^2\theta$

THEOREMS FOR DIFFERENTIATION:

Chain rule:

If y is a function of u and u is a function of x, then derivative of y with respect to x is $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

$$\text{or } y=f(u)\Rightarrow\frac{dy}{dx}=f'(u)\frac{du}{dx}$$

Sum and difference rule: Using linear property $\frac{d}{dx}(f(x)\pm g(x))=\frac{d}{dx}(f(x))\pm\frac{d}{dx}(g(x))$

Product rule: **(i)** $\frac{d}{dx}(f(x)g(x))=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$

(ii) $\frac{d}{dx}(u.v.w.)=u.v.\frac{dw}{dx}+v.w.\frac{du}{dx}+u.w.\frac{dv}{dx}$

Scalar multiple rule: $\frac{d}{dx}(k\,f(x))=k\frac{d}{dx}f(x)$

Quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d}{dx}(f(x))-f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$, provided $g(x)\neq 0$

METHODS OF DIFFERENTIATION:

Differentiation of Implicit Functions:

A function expressed in the form $f(x,y)=0$, then we say that y is an implicit function of x.

Step 1: Differentiate each term of $f(x,y)=0$ with respect to x.

Step 2: Collect the terms containing $\frac{dy}{dx}$ on one side and the terms not involving $\frac{dy}{dx}$ on the other side.

Step 3: Express $\frac{dy}{dx}$ as a function of x or y or both.

Logarithmic Differentiation:

Case I: $y = [f(x)]^{g(x)}$ where $f(x)$ and $g(x)$ are functions of x. To find the derivative of this type of functions we proceed as follows:

Let $y = [f(x)]^{g(x)}$. Taking logarithm of both the sides, we have $\log y = g(x).\log f(x)$

Differentiating with respect to x, we get $\frac{1}{y}\frac{dy}{dx} = g(x).\frac{1}{f(x)}\frac{df(x)}{dx} + \log\{f(x)\}.\frac{dg(x)}{dx}$

$$\therefore \frac{dy}{dx} = y\left[\frac{g(x)}{f(x)}.\frac{df(x)}{dx} + \log[f(x)].\frac{dg(x)}{dx}\right] = [f(x)^{g(x)}\left[\frac{g(x)}{f(x)}\frac{df(x)}{dx} + \log[f(x)\frac{dg(x)}{dx}\right]$$

Case II: $y = \frac{f_1(x).f_2(x)}{g_1(x).g_2(x)}$

Taking logarithm of both the sides, we have $\log y = \log[f_1(x)] + \log[f_2(x)] - \log[g_1(x)] - \log[g_2(x)]$

Differentiating with respect to x, we get $\frac{1}{y}\frac{dy}{dx} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} - \frac{g_1'(x)}{g_2(x)} - \frac{g_2'(x)}{g_2(x)}$

$$\frac{dy}{dx} = y\left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} - \frac{g_1'(x)}{g_1(x)} - \frac{g_2'(x)}{g_2(x)}\right] = \frac{f_1(x).f_2(x)}{g_1(x).g_2(x)}\left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} - \frac{g_1'(x)}{g_1(x)} - \frac{g_2'(x)}{g_2(x)}\right]$$

Differentiation of Parametric Functions:

If $x = \phi(t)$, $y = \psi(t)$, Then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Successive Differentiation or Higher Order Derivatives

If $y = f(x)$ then, First derivative, $\frac{dy}{dx} = \frac{d}{dx}f(x)$

Second derivative, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

ROLLE'S THEOREM:

Let f be a real valued function defined on the closed interval $[a, b]$ such that,

(1) $f(x)$ is continuous in the closed interval $[a, b]$

(2) $f(x)$ is differentiable in the open interval (a, b) and

(3) $f(a) = f(b)$

Then there is at least one value c of x in open interval (a, b) for which $f'(c) = 0$.

LAGRANGE'S MEAN VALUE THEOREM:

If a function $f(x)$,

(1) Is continuous in the closed interval $[a, b]$ and

(2) Is differentiable in the open interval (a, b)

Then there is at least one value $c \in (a, b)$, such that; $f'(c) = \frac{f(b) - f(a)}{b - a}$

APPLICATION OF DERIVATIVES

6

CHAPTER

Derivative as The Rate of Change

If y is some function of time t *i.e.,* $y = f(t)$, then small change in time Δt have a corresponding change Δy in y.

Thus, the average rate of change = $\frac{dy}{dt}$

INCREASING AND DECREASING FUNCTION

Strictly increasing function: A function $f(x)$ is said to be a strictly increasing function on (a, b), if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in (a, b).$$

Strictly decreasing function: A function $f(x)$ is said to be a strictly decreasing function on (a, b),

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in (a, b).$$

CONDITION FOR INCREASING AND DECREASING FUNCTION

Let f be a differentiable real function defined on an open interval (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).

(b) If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).

SLOPE OF THE TANGENT AND NORMAL

Slope of the tangent: If tangent is drawn on the curve $y = f(x)$ at point $P(x_1, y_1)$ and this tangent makes an angle ψ with positive x-direction then,

$$\text{Slope, } \tan\theta = \frac{dy}{dx} \text{ at } x = x_1 \text{ and } y = y_1$$

Slope of the normal: The normal to a curve at $P(x_1, y_1)$ is a line perpendicular to the tangent at P and passing through P and

$$\text{slope of the normal} = \frac{-1}{\text{Slope of tangent}} = \frac{-1}{\left(\frac{dy}{dx}\right)_{P(x_1, y_1)}} = -\left(\frac{dx}{dy}\right)_{P(x_1, y_1)}$$

EQUATION OF THE TANGENT AND NORMAL

Equation of the tangent:

The equation of the tangent to the curve $y = f(x)$ at point $P(x_1, y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Equation of the normal:

Thus, equation of the normal to the curve $y = f(x)$ at point $P(x_1, y_1)$

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$$

APPROXIMATIONS:

In this section, we will use differentials to approximate values of certain quantities.

Let $y = f(x)$ be a real valued function. Let Δx denote small increment in x and increment in y corresponding to the increment in x, denoted by Δy, is given by

$$\Delta y = f(x + \Delta x) - f(x) \qquad \ldots\text{(i)}$$

To find approximate value of $f(x + \Delta x)$,

From (i), $\quad f(x + \Delta x) = \Delta y + f(x) = dy + y$

Where $dy = \left(\frac{dy}{dx}\right) \Delta x$

MAXIMA AND MINIMA OF A FUNCTION

Conditions for Maxima and Minima of a Function

(i) The value of the function $f(x)$ at $x = a$ is maximum, if $f'(a) = 0$ and $f''(a) < 0$.

(ii) The value of the function $f(x)$ at $x = a$ is minimum if $f'(a) = 0$ and $f''(a) > 0$.

INTEGRALS

7 CHAPTER

Definition:

$$\frac{d}{dx}(\phi(x)+c)=f(x)\Rightarrow\int f(x)dx=\phi(x)+c$$

Differentiation and integration are inverse of each other.

Properties of Integrals:

(1) $\frac{d}{dx}\left[\int f(x)dx\right]=f(x)$.

(2) $\int cf(x)dx=c\int f(x)dx$.

(3) $\int\{f_1(x)\pm f_2(x)\}dx=\int f_1(x)dx\pm\int f_2(x)dx$

In the general form,

$$\int\{k_1 . f_1(x)\pm k_2 . f_2(x)\pm k_3 . f_3(x)\pm .....\}dx=k_1\int f_1(x)dx\pm k_2\int f_2(x)dx\pm k_3\int f_3(x)dx\pm ....$$

FUNDAMENTAL INTEGRATION FORMULAE

(1) **(i)** $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$ **(ii)** $\int dx=x+c$

(iii) $\int\frac{1}{\sqrt{x}}dx=2\sqrt{x}+c,$ **(iv)** $\int(ax+b)^n dx=\frac{1}{a}.\frac{(ax+b)^{n+1}}{n+1}+c$

(2) **(i)** $\int\frac{1}{x}dx=\log|x|+c$ **(ii)** $\int\frac{1}{ax+b}dx=\frac{1}{a}(\log|ax+b|+c$

(3) $\int e^x dx=e^x+c$ **(4)** $\int a^x dx=\frac{a^x}{\log_e a}+c$

(5) $\int\sin x\,dx=-\cos x+c$ **(6)** $\int\cos x\,dx=\sin x+c$

(7) $\int\sec^2 x\,dx=\tan x+c$ **(8)** $\int\text{cosec}^2 x\,dx=-\cot x+c$

(9) $\int\sec x\tan x\,dx=\sec x+c$ **(10)** $\int\text{cosec}\,x\cot x\,dx=-\text{cosec}\,x+c$

(11) $\int\tan x\,dx=-\log|\cos x|+c=\log|\sec x|+c$

(12) $\int\cot x\,dx=\log|\sin x|+c=-\log|\text{cosec}\,x|+c$

(13) $\int\sec x\,dx=\log|\sec x+\tan x|+c=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$

(14) $\int\text{cosec}\,x\,dx=\log|\text{cosec}\,x-\cot x|+c=\log\tan\frac{x}{2}+c$

(15) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c = -\cos^{-1} x + c$

(16) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c = -\cos^{-1} \frac{x}{a} + c$

(17) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$

(18) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1} \frac{x}{a} + c = \frac{-1}{a}\cot^{-1} \frac{x}{a} + c$

(19) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\mathrm{cosec}^{-1} x + c$

(20) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1} \frac{x}{a} + c = \frac{-1}{a}\mathrm{cosec}^{-1} \frac{x}{a} + c$

(21) $\int \frac{1}{x^2-a^2} = \frac{-1}{a}\coth^{-1} \frac{x}{a} + c = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$, when $x > a$

(22) $\int \frac{1}{a^2-x^2} dx = \frac{1}{a}\tanh^{-1} \frac{x}{a} + c = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$, when $x < a$

(23) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\{| x + \sqrt{x^2-a^2} |\} + c = \cosh^{-1}\left(\frac{x}{a}\right) + c$

(24) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\{| x + \sqrt{x^2+a^2} |\} + c = \sinh^{-1}\left(\frac{x}{a}\right) + c$

(25) $\int \sqrt{a^2-x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$

(26) $\int \sqrt{x^2-a^2}\, dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2 \log\{x + \sqrt{x^2-a^2}\} + c = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2 \cosh^{-1}\left(\frac{x}{a}\right) + c$

(27) $\int \sqrt{x^2+a^2}\, dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2 \log\{x + \sqrt{x^2+a^2}\} + c = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2 \sinh^{-1}\left(\frac{x}{a}\right) + c$

INTEGRATION BY SUBSTITUTION

1. $\int f[\phi(x)]\phi'(x)\,dx$ **:** Here, we put $\phi(x) = t$, so that $\phi'(x)dx = dt$ and in that case the integrand is reduced to $\int f(t)dt$. In this method, the integrand is broken into two factors so that one factor can be expressed in terms of the function whose differential coefficient is the second factor.

2. $I = \int f'(x).\, f(x).\, dx = \frac{f(x)^2}{2} + c$

3. $\frac{f'(x)}{f(x)} dx = \log[\, f(x)] + c$

4. $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c \quad [n \neq -1]$

5. $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

STANDARD SUBSTITUTIONS:

	Integrand form	Substitution
(i)	$\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$, a^2-x^2	$x=a\sin\theta$, $x=a\cos\theta$
(ii)	$\sqrt{x^2+a^2}$, $\frac{1}{\sqrt{x^2+a^2}}$, x^2+a^2	$x=a\tan\theta$ or $x=a\sinh\theta$
(iii)	$\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$, x^2-a^2	$x=a\sec\theta$ or $x=a\cosh\theta$
(iv)	$\sqrt{\frac{x}{a+x}}$, $\sqrt{\frac{a+x}{x}}$, $\sqrt{x(a+x)}$, $\frac{1}{\sqrt{x(a+x)}}$	$x=a\tan^2\theta$
(v)	$\sqrt{\frac{x}{a-x}}$, $\sqrt{\frac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\frac{1}{\sqrt{x(a-x)}}$	$x=a\sin^2\theta$
(vi)	$\sqrt{\frac{x}{x-a}}$, $\sqrt{\frac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\frac{1}{\sqrt{x(x-a)}}$	$x=a\sec^2\theta$
(vii)	$\sqrt{\frac{a-x}{a+x}}$, $\sqrt{\frac{a+x}{a-x}}$	$x=a\cos 2\theta$

INTEGRATION BY PARTS

If u and v are two functions of x, then $\int \underset{I}{u}\,\underset{II}{v}\,dx = u\int v\,dx - \int \{\frac{du}{dx}.\int v dx\}dx$

i.e., the integral of the product of two functions = (First function) × (Integral of second function) – Integral of {(Differentiation of first function) × (Integral of second function)}

EVALUATION OF THE VARIOUS FORMS OF INTEGRALS BY USE OF STANDARD RESULTS

1. Integrals of the form $\int \frac{dx}{ax^2+bx+c}$,

(i) Make the coefficient of x^2 unity by taking 'a' common from ax^2+bx+c.

(ii) Express the terms containing x^2 and x in the form of a perfect square by adding and subtracting the square of half of the coefficient of x.

(iii) Put the linear expression in x equal to t and express the integrals in terms of t.

(iv) The resultant integrand will be either in $\int \frac{dx}{x^2+a^2}$ or $\int \frac{dx}{x^2-a^2}$ or $\int \frac{dx}{a^2-x^2}$

2. Integral of the form $\int \frac{px+q}{ax^2+bx+c}dx$:

Here we express the numerator: $px+q$ as

$$px+q = M\frac{d}{dx}(ax^2+bx+c)+N$$

Replace $px+q$ in the numerator of integral and split the integral and integrate it

3. Integral of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$:

To evaluate this form of integrals, proceed as follows:

(i) Make the coefficient of x^2 unity by taking $\sqrt{a}$ common from $\sqrt{ax^2 + bx + c}$.

Then, $\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}}}$.

(ii) Put $x^2 + \frac{b}{a}x + \frac{c}{a}$, by the method of completing the square in the form, $\sqrt{A^2 - X^2}$ or $\sqrt{X^2 + A^2}$ or $\sqrt{X^2 - A^2}$ where, X is a linear function of x and A is a constant.

(iii) After this, use any of the following standard formulae according to the case under consideration

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c \Rightarrow \int \frac{dx}{\sqrt{x^2 + a^2}} = \log| x + \sqrt{x^2 + a^2} | + c$ and

$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log| x + \sqrt{x^2 - a^2} | + c.$

4. Integral of the form $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$:

To evaluate this form of integrals, first we write,

$$px + q = M\frac{d}{dx}(ax^2 + bx + c) + N \Rightarrow px + q = M(2ax + b) + N$$

Substitute the numerator and split the integral and ingetrate it

5. Integrals of the form $\int \frac{f(x)}{ax^2 + bx + c} dx$, where $f(x)$ is a polynomial of degree ≥ 2:

To evaluate the integrals of the above form, divide the numerator by the denominator. Then, the integrals take the form given by

$$\frac{f(x)}{ax^2 + bx + c} = Q(x) + \frac{R(x)}{ax^2 + bx + c} dx$$

Then, we have $\int \frac{f(x)}{ax^2 + bx + c} dx = \int Q(x)dx + \int \frac{R(x)}{ax^2 + bx + c} dx$

The integrals on R.H.S. can be obtained by the methods discussed earlier.

6. Integrals of the forms $\int \sqrt{ax^2 + bx + c}\, dx$:

To evaluate this form of integrals, express $ax^2 + bx + c$ in the form $a\left[(x + \alpha)^2 + \beta^2\right]$ by the method of completing the square and apply the standard result discussed in the above section according to the case as may be.

INTEGRATION OF RATIONAL FUNCTIONS BY USING PARTIAL FRACTIONS:

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
Where x^2+bx+c cannot be factorised further	

Above Partial fractions can be used to evaluate integrals of the form given in first column of the table. Find the constants A, B, C (By solving or by equating) and substitute and integrate.

DEFINITE INTEGRALS:

Let $\int f(x)\,dx = \phi(x)$ Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)dx$ and is defined $\int_a^b f(x)dx = \phi(b)-\phi(a)$.

DEFINITE INTEGRAL AS THE LIMIT OF A SUM:

Let $f(x)$ be a single valued continuous function defined in the interval $a \le x \le b$, where a and b are both finite. Let this interval be divided into n equal sub-intervals, each of width h by inserting $(n-1)$ points $a+h, a+2h, a+3h ...... a+(n-1)h$ between a and b. Then $nh = b-a$.

The sum $\lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$, if it exists is called the **definite integral** of $f(x)$ with respect to x between the limits a and b and we denote it by the symbol $\int_a^b f(x)dx$.

Thus, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+......+f\{a+(n-1)h\}]$

where, $nh = b-a$, a and b being the limits of integration.

EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

When the variable in a definite integral is changed, the substitutions in terms of new variable should be affected at three places.

(i) In the integrand (ii) In the differential say, dx (iii) In the limits

For example, if we put $\phi(x)=t$ in the integral $\int_a^b f\{\phi(x)\}\phi'(x)dx$, then

$$\int_a^b f\{\phi(x)\}\phi'(x)dx = \int_{\phi(a)}^{\phi(b)} f(t)\,dt .$$

PROPERTIES OF DEFINITE INTEGRAL

(1) $\int_a^b f(x)dx = \int_a^b f(t)dt$ *i.e.*, The value of a definite integral remains unchanged if its

(2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ *i.e.*, by the interchange in the limits of definite integral, the sign of the integral is changed.

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, (where $a < c < b$)

or $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_n}^b f(x)dx$; (where $a < c_1 < c_2 < .... c_n < b$)

(4) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$: This property can be used only when lower limit is zero. It is generally used for those complicated integrals whose denominators are unchanged when x is replaced by $(a - x)$.

(5) $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even function or } f(-x) = f(x) \\ 0 \quad , \text{if } f(x) \text{ is odd is odd function or } f(-x) = -f(x) \end{cases}$

This property is generally used when integrand is either even or odd function of x.

(6) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)\,dx$

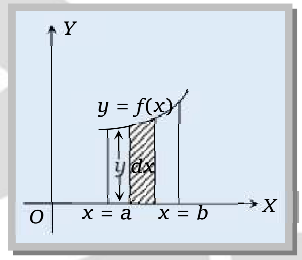
In particular, $\int_0^{2a} f(x)dx = \begin{cases} 0 \quad , \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \end{cases}$

(7) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

APPLICATION OF INTEGRALS 8
CHAPTER

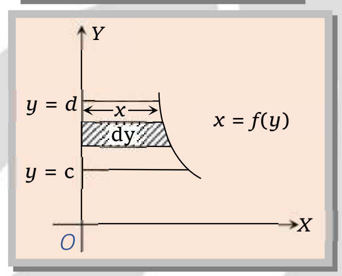
AREA OF BOUNDED REGIONS

The area bounded by a cartesian curve $y = f(x)$, x-axis and ordinates $x = a$ and $x = b$ is given by



$$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)dx$$

If the curve $y = f(x)$ lies below x-axis, then the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is negative. So, area is given by $\left|\int_a^b y\,dx\right|$



The area bounded by a cartesian curve $x = f(y)$, y-axis and abscissa $y = c$ and $y = d$ is given by

$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)dy$$

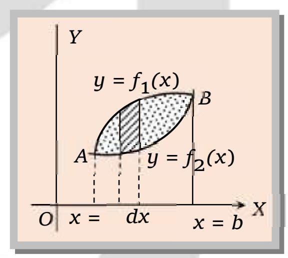
SYMMETRICAL AREA

If the curve is symmetrical about a coordinate axis (or a line or origin), then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the required area.

AREA BETWEEN TWO CURVES

(1) When both curves intersect at two points and their common area lies between these points:

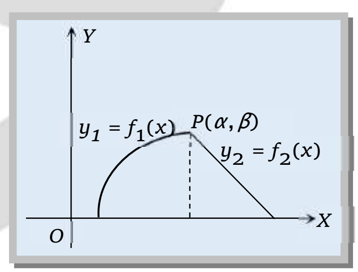
If the curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$, where $f_1(x) > f_2(x)$ intersect in two points $A(x = a)$ and $B(x = b)$, then common area between the curves is $= \int_a^b (y_1 - y_2)\,dx$



$$= \int_a^b [f_1(x) - f_2(x)]\,dx$$

(2) When two curves intersect at a point and the area between them is bounded by x-axis:

Area bounded by the curves $y = f_1(x), y_2 = f_2(x)$ and $x -$ axis is



$$= \int_a^\alpha f_1(x)dx + \int_\alpha^b f_2(x)dx$$

Where $P(\alpha, \beta)$ is the point of intersection of the two curves.

(3) Positive and negative area: Area is always taken as positive. If some part of the area lies above the x-axis and some part lies below x-axis, then the area of two parts should be calculated separately and then add their numerical values to get the desired area.

DIFFERENTIAL EQUATIONS 9

CHAPTER

Definition:

An equation involving independent variable x, dependent variable y and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, ...$ is called differential equation.

Order of a Differential Equation:

The order of a differential equation is the order of the highest derivative occurring in the differential equation.

Degree of a Differential Equation:

The degree of a differential equation is the power of the highest order derivative occurring in differential equation when it is written as a polynomial in differential coefficients.

Formation of Differential Equation:

Let the D.E has n arbitrary constants.

Step (i): Differentiate the relation in step (i) n times with respect to x.

Step (iv): Eliminate arbitrary constants with the help of n equations involving differential coefficients. The equation so obtained is the desired differential equation.

General and Particular Solution:

General Solution: The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation.

Particular Solution: Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE:

A differential equation of first order and first degree involves x, y and $\frac{dy}{dx}$. So it can be put in any one of the following forms: $\frac{dy}{dx} = f(x,y)$ or $f\left(x, y, \frac{dy}{dx}\right) = 0$ or $f(x,y)dx + g(x,y)dy = 0$ where $f(x,y)$ and $g(x,y)$ are obviously the functions of x and y.

VARIABLE SEPARABLE TYPE DIFFERENTIAL EQUATION:

A first order and first degree differential equation can be written as

$$f(x, y)dx + g(x, y)dy = 0$$

$$\text{or } \frac{dy}{dx} = \frac{f(x,y)}{g(x,y)'} \text{ or } \frac{dy}{dx} = \phi(x,y)$$

Where $f(x,y)$ and $g(x,y)$ are obviously the functions of x and y. It is not always possible to solve this type of equations. The solution of this type of differential equations is possible only when it falls under the category of some standard forms.

1. Differential equation of the form $f_1(x)dx = f_2(y)dy$:

Integrating both sides, we get its solution as $\int f_1(x)dx = \int f_2(y)dy + C$,

2. Differential equations of the type $\frac{dy}{dx} = f(x)$:

$$\frac{dy}{dx} = f(x) \Leftrightarrow dy = f(x)dx \Rightarrow \int dy = \int f(x)dx + C \text{ or } y = \int f(x)dx + C.$$

3. Differential equations of the type $\frac{dy}{dx} = f(y)$

$$\frac{dy}{dx} = f(y) \Rightarrow \frac{dx}{dy} = \frac{1}{f(y)} \Rightarrow dx = \frac{1}{f(y)}dy \Rightarrow \int dx = \int \frac{1}{f(y)}dy + C \text{ or } x = \int \frac{1}{f(y)}dy + C.$$

HOMOGENEOUS DIFFERENTIAL EQUATION:

A homogeneous differential equation means, The given differential equation can be written as $\frac{dy}{dx} = \frac{x^n f(y/x)}{x^n g(y/x)} = \frac{f(y/x)}{g(y/x)} = F\left(\frac{y}{x}\right)$.

Step (i): Put the differential equation in the form $\frac{dy}{dx} = \frac{\varphi(x,y)}{\psi(x,y)}$

Step (ii): Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the equation convert it into V.S. form and Integrate both sides to obtain the solution in terms of v and x.

Step (iii): Replace v by $\frac{y}{x}$ in the solution to obtain the solution in terms of x and y.

LINEAR DIFFERENTIAL EQUATION:

1. Form $\frac{dy}{dx} + Py = Q$: Where P and Q are functions of x (or constants)

Find integrating factor (I.F.) given by $I.F. = e^{\int Pdx}$.

The solution is given by integrating: $y(I.F.) = \int Q(I.F.)dx + C$

2. Form $\frac{dx}{dy} + Rx = S$: where R and S are functions of y or constants. Note that y is independent variable and x is a dependent variable.

Algorithm for solving linear differential equations of the form $\frac{dx}{dy} + Rx = S$

Find I.F. by using $I.F. = e^{\int R\,dy}$

The solution is given by integrating: $x(I.F.) = \int S(I.F.)dy + C$

VECTOR ALGEBRA

10

CHAPTER

Scalar and vector quantities:

Scalars are Those quantities which have only magnitude and which are not related to any fixed direction in space.

Vectors are those which have both magnitude and direction.

Representation of Vectors:

Geometrically a vector is represented by a line segment. For example, $\boldsymbol{a} = \overrightarrow{AB}$. Here A is called the initial point and B, the terminal point or tip.

Magnitude or modulus of $\boldsymbol{a}$ is expressed as $|\boldsymbol{a}| = |\overrightarrow{AB}| = AB$.

Types of Vectors:

Zero or null vector: A vector whose magnitude is zero is called zero or null vector and it is represented by $\vec{O}$.

Unit vector: A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector $\boldsymbol{a}$ is denoted by$\hat{\boldsymbol{a}}$, read as "*a cap*". Thus, $|\hat{\boldsymbol{a}}| = 1$.

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{\text{Vector } a}{\text{Magnitude of } a}$$

Like and unlike vectors: Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.

Collinear or parallel vectors: Vectors having the same or parallel supports are called collinear vectors.

Co-initial vectors: Vectors having the same initial point are called *co-initial vectors*.

Co-planar vectors: A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

Negative of a vector: The vector which has the same magnitude as the vector $\boldsymbol{a}$ but opposite direction, is called the negative of $\boldsymbol{a}$ and is denoted by $-\boldsymbol{a}$. Thus, if $\overrightarrow{PQ} = \boldsymbol{a}$, then $\overrightarrow{QP} = -\boldsymbol{a}$.

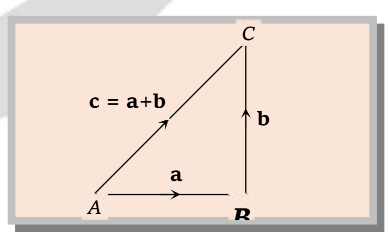
Position vectors: The vector $\overrightarrow{OA}$which represents the position of the point A with respect to a fixed point O (called origin) is called position vector of the point A. If (x, y, z) are co-ordinates of the point A, then

$$\overrightarrow{OA} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

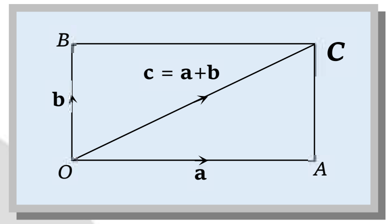
Equality of vectors: Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be equal, if Their magnitude and directions are same

Addition of Vectors

Triangle law of addition: If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle, but in opposite direction. This is known as the triangle law of addition of vectors. Thus, if $|\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a} - \boldsymbol{b}| \Rightarrow \boldsymbol{a} \perp \boldsymbol{b}$ and $\overrightarrow{AC} = \boldsymbol{c}$ then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ *i.e.,* $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$.



Parallelogram law of addition: If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as parallelogram law of addition of vectors.



Thus, if $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$

Then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ *i.e.,* $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$,where *OC* is a diagonal of the parallelogram *OABC*.

Addition in component form : If the vectors are defined as $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then their sum is defined as $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$.

Properties of vector addition:

(a) Binary operation: The sum of two vectors is always a vector.

(b) Commutativity: For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$

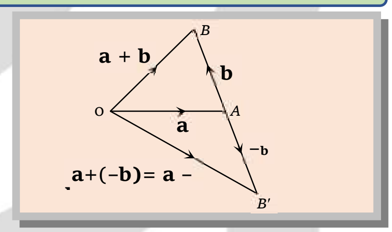
(c) Associativity: For any three vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$, $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$

(d) Identity: Zero vector is the identity for addition. For any vector $\boldsymbol{a}$, $|\boldsymbol{a}| = 3, |\boldsymbol{b}| = 4$

(e) Additive inverse: For every vector $\boldsymbol{a}$ its negative vector $-\boldsymbol{a}$ exists such that $\boldsymbol{a} + (-\boldsymbol{a}) = (-\boldsymbol{a}) + \boldsymbol{a} = \boldsymbol{0}$ *i.e.,* $(-\boldsymbol{a})$ is the additive inverse of the vector **a**.

Subtraction of vectors:

If $\boldsymbol{v}$ and $\boldsymbol{b}$ are two vectors, then their subtraction $\boldsymbol{a} - \boldsymbol{b}$ is defined as $\boldsymbol{w}$ where $-\boldsymbol{b}$ is the negative of $\boldsymbol{b}$ having magnitude equal to that of $\boldsymbol{b}$ and direction opposite to $\boldsymbol{b}$.



If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

Then $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$.

Multiplication of a vector by a scalar:

If $\boldsymbol{a}$ is a vector and *m* is a scalar (*i.e.,* a real number) then $m\boldsymbol{a}$ is a vector whose magnitude is *m* times that of $\boldsymbol{a}$ and whose direction is the same as that of $\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

∴ Magnitude of $m\boldsymbol{a} = |m\boldsymbol{a}| \Rightarrow$ *m* (magnitude of $\boldsymbol{a}$) = $m|\boldsymbol{a}|$

Again if $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ then $m\boldsymbol{a} = (ma_1)\boldsymbol{i} + (ma_2)\boldsymbol{j} + (ma_3)\boldsymbol{k}$

Properties of Multiplication of vectors by a scalar : The following are properties of multiplication of vectors by scalars, for vectors $\boldsymbol{a}, \boldsymbol{b}$ and scalars *m*, *n*

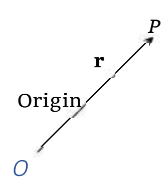
(i) $m(-\boldsymbol{a}) = (-m)\boldsymbol{a} = -(m\boldsymbol{a})$ (ii) $(-m)(-\boldsymbol{a}) = m\boldsymbol{a}$

(iii) $m(n\boldsymbol{a}) = (mn)\boldsymbol{a} = n(m\boldsymbol{a})$ (iv) $(m + n)\boldsymbol{a} = m\boldsymbol{a} + n\boldsymbol{a}$

(v) $m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$

Position Vector

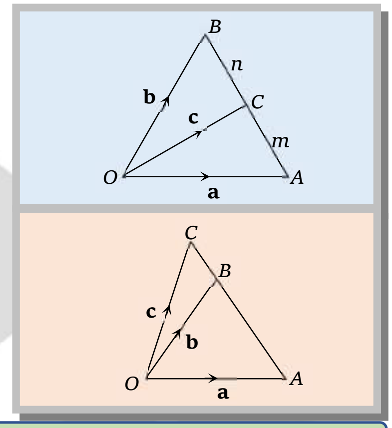
If a point *O* is fixed as the origin in space (or plane) and *P* is any point, then $\overrightarrow{OP}$ is called the position vector of *P* with respect to *O*.



For any Vector $\overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ =(Position vector of *B*) – (Position vector of *A*)

SECTION FORMULA



(i) Internal division: Let A and B be two points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, and let C be a point dividing AB internally in the ratio $m : n$.
Then the position vector of C is given by

$$\overrightarrow{OC} = \frac{m\boldsymbol{b}+n\boldsymbol{a}}{m+n}$$

(ii) External division: Let A and B be two points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively and let C be a point dividing AB externally in the ratio $m : n$.
Then the position vector of C is given by

$$\overrightarrow{OC} = \frac{m\boldsymbol{b}-n\boldsymbol{a}}{m-n}$$

Relation Between Two Parallel Vectors

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ then $\boldsymbol{a}||\boldsymbol{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Scalar or Dot Product of two Vectors:

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by $\boldsymbol{a}.\boldsymbol{b}$ and is defined as the scalar $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$, where $|\boldsymbol{a}|$ and $|\mathbf{b}|$ are modulus of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Angle between two vectors: If $\boldsymbol{a}, \boldsymbol{b}$ be two vectors inclined at an angle θ, then,

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

Properties of Scalar Product

(i) Commutativity: The scalar product of two vector is commutative

i.e., $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$.

(ii) Distributivity of scalar product over vector addition: The scalar product of vectors is distributive over vector addition *i.e.,*

(a) $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$ (Left distributivity)

(b) $(\boldsymbol{b} + \boldsymbol{c}) \cdot \boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{a}$ (Right distributivity)

(iii) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors $\boldsymbol{a}.\boldsymbol{b} = 0 \Leftrightarrow \boldsymbol{a} \perp \boldsymbol{b}$.

$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{i} = 0$; $\boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{j} = 0$; $\boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{i} \cdot \boldsymbol{k} = 0$.

(iv) For any vector $\boldsymbol{a}$, $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.

As $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit vectors along the co-ordinate axes, therefore $\boldsymbol{i} \cdot \boldsymbol{i} = |\boldsymbol{i}|^2 = 1$,
$\boldsymbol{j} \cdot \boldsymbol{j} = |\boldsymbol{j}|^2 = 1$ and $\boldsymbol{k} \cdot \boldsymbol{k} = |\boldsymbol{k}|^2 = 1$

(v) If m is a scalar and $\boldsymbol{a}, \boldsymbol{b}$ be any two vectors, then $(m\boldsymbol{a}).\boldsymbol{b} = m(\boldsymbol{a}.\boldsymbol{b}) = \boldsymbol{a}.(m\boldsymbol{b})$

Scalar product in terms of components:

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then, $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$. Thus, scalar product of two vectors is equal to the sum of the products of their corresponding components. In particular, $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}|^2 = a_1^2 + a_2^2 + a_3^2$.

Vector or cross product of two vectors:

Let $\boldsymbol{a}, \boldsymbol{b}$ be two non-zero, non-parallel vectors. Then the vector product $\boldsymbol{a} \times \boldsymbol{b}$, is defined as,
$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\,\hat{\boldsymbol{\eta}}$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, $\hat{\boldsymbol{\eta}}$ is a unit vector perpendicular to the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{\eta}}$ form a right handed system.

Properties of Vector Product

(i) Vector product is not commutative *i.e.,* if $\boldsymbol{a}$ and $\boldsymbol{b}$ are any two vectors, then

$$\boldsymbol{a}\times\boldsymbol{b}\neq\boldsymbol{b}\times\boldsymbol{a}, \text{ however, } \boldsymbol{a}\times\boldsymbol{b}=-(\boldsymbol{b}\times\boldsymbol{a})$$

(ii) Distributivity of vector product over vector addition.

Let $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ be any three vectors. Then

(a) $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}$ (Left distributivity)

(b) $(\boldsymbol{b}+\boldsymbol{c})\times\boldsymbol{a}=\boldsymbol{b}\times\boldsymbol{a}+\boldsymbol{c}\times\boldsymbol{a}$ (Right distributivity)

(iii) The vector product of two non-zero vectors is zero vector *iff* they are parallel (Collinear) *i.e.,* $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}\Leftrightarrow\boldsymbol{a}||\boldsymbol{b},\boldsymbol{a},\boldsymbol{b}$ are non-zero vectors.

(iv) It follows from the above property that $\boldsymbol{a}\times\boldsymbol{a}=\boldsymbol{0}$ for every non-zero vector $\boldsymbol{a}$, which in turn implies that $\boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{j}\times\boldsymbol{j}=\boldsymbol{k}\times\boldsymbol{k}=\boldsymbol{0}$

(vi) Vector product of orthonormal triad of unit vectors **i, j, k** using the definition of the vector product, we obtain $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k},\boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i},\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j},\boldsymbol{j}\times\boldsymbol{i}=-\boldsymbol{k},\boldsymbol{k}\times\boldsymbol{j}=-\boldsymbol{i},\boldsymbol{i}\times\boldsymbol{k}=-\boldsymbol{j}$

Vector Product in terms of Components:

If $\boldsymbol{a}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$ and $\boldsymbol{b}=b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k}$.

$$\text{Then, } \boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

Area of Parallelogram and Triangle:

(i) The area of a parallelogram with adjacent sides $\boldsymbol{a}$ and $\boldsymbol{b}$ is $|\boldsymbol{a}\times\boldsymbol{b}|$.

(ii) The area of a parallelogram with diagonals $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$.

Scalar triple product:

If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ are three vectors, then their scalar triple product is denoted by $[\boldsymbol{abc}]$ and defined as

$$[\boldsymbol{abc}]=\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$$

Geometrical interpretation of scalar triple product:

The scalar triple product of three vectors is equal to the volume of the parallelopiped whose three coterminous edges are represented by the given vectors. $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ form a right handed system of vectors. Therefore

$[\boldsymbol{abc}]=(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c}=$ volume of the parallelopiped, whose coterminous edges are **a**, **b** and **c**.

Properties of scalar triple product:

(i) If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ are cyclically permuted, the value of scalar triple product remains the same. *i.e.,* $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c}=(\boldsymbol{b}\times\boldsymbol{c}).\boldsymbol{a}=(\boldsymbol{c}\times\boldsymbol{a}).\boldsymbol{b}$ or $[\boldsymbol{abc}]=[\boldsymbol{bca}]=[\boldsymbol{cab}]$

(ii) The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude *i.e.,* $[\boldsymbol{abc}]=-[\boldsymbol{bac}]=-[\boldsymbol{cba}]=-[\boldsymbol{acb}]$

(iii) In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same *i.e.,* $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c}=\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$

Scalar triple product in terms of components

(i) If $\boldsymbol{a}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$, $\boldsymbol{b}=b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k}$ and $\boldsymbol{c}=c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k}$ be three vectors.

$$\text{Then, } [\boldsymbol{abc}]=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$$

THREE-DIMENSIONAL GEOMETRY 11

CHAPTER

Distance formula: The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

Distance from origin: Let O be the origin and $P(x, y, z)$ be any point, then

$OP = \sqrt{(x^2 + y^2 + z^2)}$.

Section formula for internal division: Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Let R be a point on the line segment joining P and Q such that it divides the join of P and Q internally in the ratio $m_1 : m_2$. Then the co-ordinates of R are $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$.

Section formula for external division:

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points, and let R be a point on PQ produced, dividing it externally in the ratio $m_1 : m_2$ $(m_1 \neq m_2)$. Then the co-ordinates of R are $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$.

Co-ordinates of the midpoint:

When division point is the mid-point of PQ then ratio will be 1 : 1, hence co-ordinates of the mid point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Co-ordinates of the general point:

The co-ordinates of any point lying on the line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as

$$\left(\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1}, \frac{kz_2 + z_1}{k + 1}\right)$$

which divides PQ in the ratio $k : 1$. This is called general point on the line PQ.

Co-ordinates of the centroid of a triangle:

If $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) are the vertices of a triangle, then co-ordinates of its centroid are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

Direction Cosines:

If α, β, γ be the angles which a given directed line makes with the positive direction of the x, y, z co-ordinate axes respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the given line and are generally denoted by l, m, n respectively.

Thus, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$

Relation between the direction cosines:

$x = lr, y = mr$ and $z = nr$

$l^2 + m^2 + n^2 = 1$

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Direction ratio:

If a, b, c are three numbers proportional to direction cosines l, m, n of a line, then a, b, c are called its direction ratios.

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \; m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \; n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

D.C.'s and D.R.'s of a line joining two points: The direction ratios of line PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1 = a$, $y_2 - y_1 = b$ and $z_2 - z_1 = c$ (say).

Then direction cosines are,

$$l = \frac{x_2 - x_1}{PQ}, m = \frac{y_2 - y_1}{PQ}, n = \frac{z_2 - z_1}{PQ}.$$

Angle Between Two Lines:

Let θ be the angle between two straight then,

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2 \text{ or } \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}$$

<u>Condition of perpendicularity:</u> If the given lines are perpendicular, then

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \text{ or } a_1a_2 + b_1b_2 + c_1c_2 = 0$$

<u>Condition of parallelism:</u> If the given lines are parallel,

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

THE STRAIGHT LINE

<u>Equation of a Line Passing Through a Given Point:</u>

Cartesian Form: Equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Equation of a line passing through (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

The co-ordinates of any point on the line are $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$

Vector form: Vector equation of a straight line passing through a fixed point with position vector **a** and parallel to a given vector **b** is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}.$$

Equation of Line Passing Through Two Given Points:

Cartesian form: If $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ be two given points, the equations to the line AB is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

The co-ordinates of any point on the line are $x=\frac{\lambda x_2+x_1}{\lambda+1}, y=\frac{\lambda y_2+y_1}{\lambda+1}, z=\frac{\lambda z_2+z_1}{\lambda+1}$

Vector form: The vector equation of a line passing through two points with position vectors **a** and **b** is

$$\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$$

Angle Between Two Lines:

Angle between two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Condition of perpendicularity: If the lines are perpendicular, then

$$a_1a_2+b_1b_2+c_1c_2=0$$

Condition of parallelism: If the lines are parallel, then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

Shortest Distance Between Two Straight Lines

1. Shortest distance between two skew lines:

Cartesian form: Shortest distance between two skew lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ is given by

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-l_1n_2)^2+(l_1m_2-l_2m_1)^2}}$$

Vector form: Let l_1 and l_2 be two lines whose equations are $l_1:\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $l_2:\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}_2$ respectively. Then, Shortest distance

$$PQ=\left|\frac{(\mathbf{b}_1\times\mathbf{b}_2).(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|=\frac{|[\mathbf{b}_1\,\mathbf{b}_2\,(\mathbf{a}_2-\mathbf{a}_1)]}{|\mathbf{b}_1\times\mathbf{b}_2|}$$

2. Shortest distance between two parallel lines:

The shortest distance between the parallel lines $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}$ and $\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}$ is given by

$d=\frac{|(\mathbf{a}_2-\mathbf{a}_1)\times\mathbf{b}|}{|\mathbf{b}|}$.

Condition for two lines to be intersecting *i.e.* coplanar:

Cartesian form: If the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ intersect, then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

Vector form : The lines $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $\mathbf{r}=\mathbf{a}_2+\lambda\mathbf{b}_2$ intersect, if $(\mathbf{a}_2-\mathbf{a}_1).(\mathbf{b}_1\times\mathbf{b}_2)=0$

THE PLANE:

General equation of plane:

Every equation of first degree of the form $Ax + By + Cz + D = 0$ represents the equation of a plane. The coefficients of x, y and z *i.e.* A, B, C are the direction ratios of the normal to the plane.

Equation of co-ordinate planes:

XOY -plane: $z = 0$

YOZ -plane: $x = 0$

ZOX -plane: $y = 0$

Equation of plane:

Vector Form: Vector equation of a plane through the point $A(\mathbf{a})$ and perpendicular to the vector **n** is $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ or $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

Cartesian Form: Equation of plane passing through the point (x_1, y_1, z_1) is

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, where A, B and C are d.r.'s of normal to the plane.

Normal form:

Vector equation of a plane normal to unit vector $\hat{\mathbf{n}}$ and at a distance d from the origin is $\mathbf{r}.\hat{\mathbf{n}} = d$.

Normal form of the equation of plane if l, m, n are the d.c.'s of the normal to the plane and d is the length of perpendicular from the origin is

$$lx + my + nz = d,$$

Intercept form:

If the plane cuts the intercepts of length a, b, c on co-ordinate axes, then its equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Equation of plane through the origin:

$$Ax + By + Cz = 0.$$

Equation of plane parallel to co-ordinate planes or perpendicular to co-ordinate axes:

(i) Equation of plane parallel to YOZ-plane (or perpendicular to x-axis) and at a distance 'a' from it is $x = a$.

(ii) Equation of plane parallel to ZOX-plane (or perpendicular to y-axis) and at a distance 'b' from it is $y = b$.

(iii) Equation of plane parallel to XOY-plane (or perpendicular to z-axis) and at a distance 'c' from it is $z = c$.

Equation of plane perpendicular to co-ordinate planes or parallel to co-ordinate axes:

(i) Equation of plane perpendicular to YOZ-plane or parallel to x-axis is $By + Cz + D = 0$.

(ii) Equation of plane perpendicular to ZOX-plane or parallel to y axis is $Ax + Cz + D = 0$.

(iii) Equation of plane perpendicular to XOY-plane or parallel to z-axis is $Ax + By + D = 0$.

Equation of plane passing through the intersection of two planes:

Cartesian form: Equation of plane through the intersection of two planes $P = a_1x + b_1y + c_1z + d_1 = 0$ and $Q = a_2x + b_2y + c_2z + d_2 = 0$ is $P + \lambda Q = 0$, where λ is the parameter.

Vector form : The equation of any plane through the intersection of planes $\mathbf{r}.\mathbf{n}_1 = d_1$ and $\mathbf{r}.\mathbf{n}_2 = d_2$ is $\mathbf{r}.(\mathbf{n}_1 + \lambda \mathbf{n}_2) = d_1 + \lambda d_2$, where λ is an arbitrary constant.

Equation of plane through three points:

Vector form: The equation of plane passing through three non-collinear points with Position vecors **a**, **b** and **c** is $(\mathbf{r} - \mathbf{a}) \cdot [(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})] = \mathbf{0}$

Cartesian Form: The equation of plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

Angle Between Two Planes:

Cartesian form: Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}\right)$$

If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then the planes are perpendicular to each other.

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the planes are parallel to each other.

Vector form: An angle θ between the planes $\mathbf{r}_1.\mathbf{n}_1 = d_1$ and $\mathbf{r}_2.\mathbf{n}_2 = d_2$ is given by

$$\cos\theta = \pm\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}.$$

Angle Between a Line and a Plane:

Vector form: Angle between Line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane $\mathbf{r} \cdot \mathbf{n} = \mathbf{d}$ is

$$\theta = \sin^{-1}\left(\frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}||\mathbf{n}|}\right)$$

Cartesian form: Angle between Line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and the plane $\mathrm{a}_2x + \mathrm{b}_2\mathrm{i} + \mathrm{c}_2z = \mathrm{d}$ is

$$\theta = \sin^{-1}\left(\frac{\mathrm{a}_1\mathrm{a}_2 + \mathrm{b}_1\mathrm{b}_2 + \mathrm{c}_1\mathrm{c}_2}{\sqrt{\mathrm{a}_1^2 + \mathrm{b}_1^2 + \mathrm{c}_1^2}\sqrt{\mathrm{a}_2^2 + \mathrm{b}_2^2 + \mathrm{c}_2^2}}\right)$$

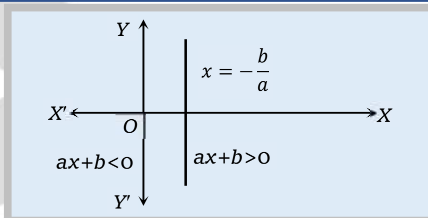
LINEAR PROGRAMMING

12
CHAPTER

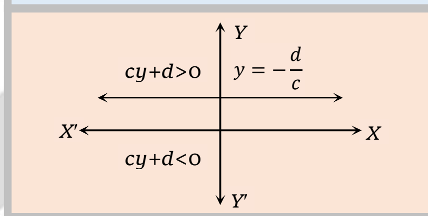
LINEAR INEQUATIONS

(i) Linear inequation in one variable:

$ax+b>0, ax+b<0, cy+d>0$ etc. are called linear inequations in one variable. Graph of these inequations can be drawn as follows:

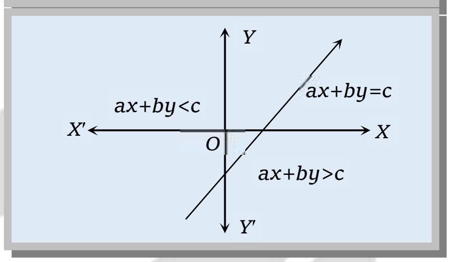


The graph of $ax+b>0$ and $ax+b<0$ are obtained by dividing *xy*-plane in two semi-planes by the line $x=-\frac{b}{a}$ (which is parallel to *y*-axis). Similarly, for $cy+d>0$ and $cy+d<0$.



(ii) Linear Inequation in two variables:

General form of these inequations are $ax+by>c, ax+by<c$. If any ordered pair (x_1, y_1) satisfies some inequations, then it is said to be a solution of the inequations.
The graph of these inequations is given below (for $c>0$):



<u>Working rule:</u>

To draw the graph of an inequation, following procedure is followed:

(i) Write the equation $ax+by=c$ in place of $ax+by<c$ and $ax+by>c$.

(ii) Make a table for the solutions of $ax+by=c$.

(iii) Now draw a line with the help of these points. This is the graph of the line $ax+by=c$.

(iv) If the inequation is > or <, then the points lying on this line is not considered and line is drawn dotted or discontinuous.

(v) If the ineuqation is $\geq$ or $\leq$, then the points lying on the line is considered and line is drawn bold or continuous.

(vi) This line divides the plane *XOY* in two region.

To Find the region that satisfies the inequation, we apply the following rules:

(a) Take an arbitrary point which will be in either region.

(b) If it satisfies the given inequation, then the required region will be the region in which the arbitrary point is located.

(c) If it does not satisfy the inequation, then the other region is the required region.

(d) Draw the lines in the required region or make it shaded.

Terms of Linear Programming

The term programming means planning and refers to a process of determining a particular program.

Objective function: The linear function which is to be optimized (maximized or minimized) is called objective function of the *L.P.P*.

Constraints or Restrictions: The conditions of the problem expressed as simultaneous equations or inequalities are called constraints or restrictions.

Non-negative Constraints: Variables applied in the objective function of a linear programming problem are always non-negative. The inequalities which represent such constraints are called non-negative constraints.

Basic variables: The m variables associated with columns of the $m \times n$ non-singular matrix which may be different from zero, are called basic variables.

Basic solution: A solution in which the vectors associated to m variables are linear and the remaining $(n-m)$ variables are zero, is called a basic solution. A basic solution is called a degenerate basic solution, if at least one of the basic variables is zero and basic solution is called non-degenerate, if none of the basic variables is zero.

Feasible solution: The set of values of the variables which satisfies the set of constraints of linear programming problem (*L.P.P*) is called a feasible solution of the *L.P.P*.

Optimal solution: A feasible solution for which the objective function is minimum

Mathematical Formulation of A Linear Programming Problem

There are mainly four steps in the mathematical formulation of a linear programming problem, as mathematical model. We will discuss formulation of those problems which involve only two variables.

(1) Identify the decision variables and assign symbols x and y to them. These decision variables are those quantities whose values we wish to determine.

(2) Identify the set of constraints and express them as linear equations/inequations in terms of the decision variables. These constraints are the given conditions.

(3) Identify the objective function and express it as a linear function of decision variables. It might take the form of maximizing profit or production or minimizing cost.

(4) Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

Graphical Solution of Two Variable Linear Programming Problem

Working rule

(i) Formulate mathematically the *L.P.P*.

(ii) Draw graph for every constraint.

(iii) Find the feasible solution region.

(iv) Find the coordinates of the vertices of feasible solution region.

(v) Calculate the value of objective function at these vertices.

(vi) Optimal value (minimum or maximum) is the required solution.

To find the Vertices of Simple Feasible Region Without Drawing A Graph

(1) Bounded region: The region surrounded by the inequalities $ax + by \leq m$ and $cx + dy \leq n$ in first quadrant is called bounded region. It is of the form of triangle or quadrilateral. Change these inequalities into equation, then by putting $x = 0$ and $y = 0,$ we get the solution also by solving the equation in which there may be the vertices of bounded region.

The maximum value of objective function lies at one vertex in limited region.

(2) Unbounded region : The region surrounded by the inequations $ax + by \geq m$ and $cx + dy \geq n$ in first quadrant, is called unbounded region.

Change the inequation in equations and solve for $x = 0$ and $y = 0$. Thus we get the vertices of feasible region.

The minimum value of objective function lies at one vertex in unbounded region but there is no existence of maximum value.

PROBLEMS HAVING INFEASIBLE SOLUTIONS

In some of the linear programming problems, constraints are inconsistent *i.e.* there does not exist any point which satisfies all the constraints. Such type of linear programming problems are said to have *infeasible solution.*

PROBABILITY

13 CHAPTER

Deterministic experiment:

Those experiments which when repeated under identical conditions produce the same result or outcome are known as deterministic experiments.

Random experiment:

If an experiment, when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes then such an experiment is known as a probabilistic experiment or a random experiment.

Sample space:

The set of all possible outcomes of a trial (random experiment) is called its sample space. It is generally denoted by *S*.

Events and Types of events:

An event is a subset of a sample space.

Equally likely events: Events are equally likely if there is no reason for an event to occur in preference to any other event.

Mutually exclusive or disjoint events: Events are said to be mutually exclusive or disjoint or incompatible if the occurrence of any one of them prevents the occurrence of all the others.

Independent events: Events are said to be independent if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of others.

Dependent events: Two or more events are said to be dependent if the happening of one event affects (partially or totally) other event.

Exhaustive number of cases: The total number of possible outcomes of a random experiment in a trial is known as the exhaustive number of cases.

Favorable number of cases: The number of cases favorable to an event in a trial is the total number of elementary events such that the occurrence of any one of them ensures the happening of the event.

Mutually exclusive and exhaustive system of event:

If $E_1, E_2, ..., E_n$ are elementary events associated with a random experiment, then

(i) $E_i \cap E_j = \phi$ for $i \neq j$ and **(ii)** $E_1 \cup E_2 \cup .... \cup E_n = S$

So, the collection of elementary events associated with a random experiment always form a system of mutually exclusive and exhaustive system of events.

Classical Definition of Probability:

If a random experiment results in *n* mutually exclusive, equally likely and exhaustive outcomes, out of which *m* are favorable to the occurrence of an event *A*, then the probability of occurrence of *A* is given by

$$P(A) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to } A}{\text{Number of total outcomes}}\text{; always } 0 \le P(A) \le 1.$$

Further, if $\overline{A}$ denotes negative of *A i.e.* event that *A* doesn't happen, then for above cases *m*, *n*; we shall have

$$P(\overline{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A) \text{ and } P(A) + P(\overline{A}) = 1.$$

Notations: For two events *A* and *B*,

(i) A' or $\overline{A}$ or A^C stands for the non-occurrence or negation of *A*.

(ii) $A \cup B$ stands for the occurrence of at least one of *A* and *B*.

(iii) $A \cap B$ stands for the simultaneous occurrence of *A* and *B*.

(iv) $A' \cap B'$ stands for the non-occurrence of both *A* and *B*.

Conditional Probability:

Probability of occurrence of *A*, given that *B* has already happened.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}.$$

Similarly, Probability of occurrence of *B*, given that *A* has already happened.

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)}.$$

Multiplication Theorems on Probability:

If *A* and *B* are two events associated with a random experiment, then $P(A \cap B) = P(A).P(B/A)$, if $P(A) \neq 0$ or $P(A \cap B) = P(B).P(A/B)$, if $P(B) \neq 0$.

<u>Extension of multiplication theorem:</u> If $A_1, A_2, ..., A_n$ are *n* events related to a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap .... \cap A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1 \cap A_2)....P(A_n/A_1 \cap A_2 \cap ... \cap A_{n-1}),$$

where $P(A_i/A_1 \cap A_2 \cap ... \cap A_{i-1})$ represents the conditional probability of the event A_i, given that the events $A_1, A_2, ..., A_{i-1}$ have already happened.

<u>Multiplication theorems for independent events:</u> If *A* and *B* are independent events associated with a random experiment, then $P(A \cap B) = P(A).P(B)$

<u>Extension of multiplication theorem for independent events:</u> If $A_1, A_2, ..., A_n$ are independent events associated with a random experiment, then $P(A_1 \cap A_2 \cap A_3 \cap ... \cap A_n) = P(A_1)P(A_2)...P(A_n)$.

The law of total probability:

Let *S* be the sample space and let $E_1, E_2, ... E_n$ be *n* mutually exclusive and exhaustive events associated with a random experiment. If *A* is any event which occurs with E_1 or E_2 or …. or E_n, then $P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + ... + P(E_n)P(A/E_n)$.

Baye's Theorem:

Let S be a sample space and $E_1, E_2, \ldots.. E_n$ be n mutually exclusive events such that $\bigcup_{i=1}^{n} E_i = S$ and $P(E_i) > 0$ for i = 1, 2, ……, n. We can think of (E_i's as the causes that lead to the outcome of an experiment. The probabilities $P(E_i)$, i = 1, 2, ….., n are called prior probabilities. Suppose the experiment results in an outcome of event A, where $P(A) > 0$. We have to find the probability that the observed event A was due to cause E_i, that is, we seek the conditional probability $P(E_i / A)$. These probabilities are called posterior probabilities, given by Baye's rule as

$$P(E_i / A) = \frac{P(E_i).P(A / E_i)}{\sum_{k=1}^{n} P(E_k)P(A / E_k)}.$$

Binomial Distribution

A random variable X which takes values 0, 1, 2, …, n is said to follow binomial distribution if its probability distribution function is given by

$$P(X = r) = {}^nC_r p^r q^{n-r}, r = 0, 1, 2, \ldots.., n$$; where $p, q > 0$ such that $p + q = 1$.

(a) Probability of Occurrence of the event exactly r times

$$P(X = r) = {}^nC_r q^{n-r} p^r.$$

(b) Probability of Occurrence of the event at least r times

$$P(X \geq r) = {}^nC_r q^{n-r} p^r + \ldots + p^n = \sum_{X=r}^{n} {}^nC_X p^X q^{n-X} \quad .$$

(c) Probability of Occurrence of the event at the most r times

$$P(0 \leq X \leq r) = q^n + {}^nC_1 q^{n-1} p + \ldots + {}^nC_r q^{n-r} p^r = \sum_{X=0}^{r} p^X q^{n-X}.$$

Mean and variance of the binomial distribution:

The mean of this distribution is np,

The variance of the Binomial distribution is $\sigma^2 = npq$ and the standard deviation is $\sigma = \sqrt{(npq)}$.